Equilibrium Selection for Markov Processes via Random Trajectory Entropy with Applications to Finite Population Biology

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- Rather than rest or accumulation points, quantities like stationary distributions and entropy rate are used to characterize long run behavior of a Markov Process
- Entropy rate also incorporates short run behavior from the entropy of the distribution of transition probabilities

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- Entropy Rate is an invariant (actually the Kolmogorov-Sinai invariant of a dynamical system)
- Can be used for equilibrium selection for a process in conjunction with random walk entropy

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$$Pr(V) = T(v_0, v_1)T(v_1, v_2) \cdots T(v_{k-1}, v_k).$$

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Since the process is irreducible, we have that the sum over all possible such trajectories from v_0 to v_k is one, forming a probability distribution. Let $\mathcal{T}(v_0, v_k)$ be the set of all such paths and define the *random trajectory entropy* (RTE) from v_0 to v_k to be the entropy of the probability distribution on $\mathcal{T}(v_0, v_k)$, i.e.

$$H_{v_0 v_k} = -\sum_{v \in \mathcal{T}(v_0, v_k)} Pr(v) \log Pr(v).$$

Random Trajectory Entropy

Let the stationary distribution of the Markov process X be given by s and the entropy rate by $H(X)$. It was shown Ekroot and Cover (Theorem 1, p.1419, 1993) that when the starting and ending states v are the same, the entropy of the random trajectory is determined by the entropy rate and the stationary probability at the state:

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For two states of the same process, we can compare the relative stability by comparing the stationary distribution values:

$$\frac{H_v}{H_w} = \frac{H(X)}{s(v)} \frac{s(w)}{H(X)} = \frac{s(w)}{s(v)}$$

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- $N = a + b$ is the total population size
- The entropy rate vanishes $\lim_{\mu \to 0} H(X) = 0$
- The limiting stationary distribution becomes a delta distribution on the two states where one of the population types dominates the population, $(0, N)$ or $(N, 0)$

Example: Moran Process

We can express the limiting stationary distribution in terms of the fixation probabilities of the two types ρ_A and ρ_B:

$$\lim_{\mu\to 0} s(0,N) = \frac{\rho_B}{\rho_A+\rho_B} \quad \text{and} \quad \lim_{\mu\to 0} s(N,0) = \frac{\rho_A}{\rho_A+\rho_B}$$

Hence we have that

$$\lim_{\mu\to 0} \frac{H_{(0,N)}}{H_{(N,0)}} = \lim_{\mu\to 0} \frac{s(N,0)}{s(0,N)} = \frac{\rho_B}{\rho_A}$$

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For the classical Moran process with game matrix $G = \left(\begin{smallmatrix} r & r \\ 1 & 1 \end{smallmatrix}\right)$

$$\lim_{\mu \to 0} \frac{H_{(0,N)}}{H_{(N,0)}} = \frac{\rho_B}{\rho_A} = \frac{1 - r^{1-N}}{1 - r^{-1}}$$

As expected, whether $r > 1$ determines which equilibrium is favored. If $r = 1$, $\rho_A = 1/N = \rho_B$ and the RTEs are equal.

Relationship to Stationary Stability for the Moran Process

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- (Harper, Fryer, 2013) The stationary local maxima of the Moran process with mutation capture the evolutionarily stable states (e.g. for the replicator equation)
- As various parameters (mutation rate μ, strength of selection β, and population size N) change so does the entropy rate and the stationary distribution
- The random trajectory entropy allows us to compare stability of equilibria as the parameters vary

Example: Hawk-Dove Game

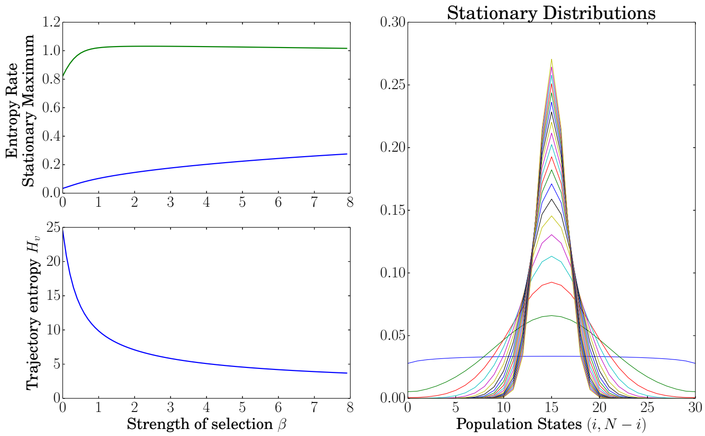

Figure : Right: Stationary distributions for Hawk-Dove landscapes for varying strength of selection $\beta \in [0, 10]$, $N = 30$, $\mu = 1/N$. Upper Left: As β increases, so does the stationary probability (blue, lower curve) of the maxima at $(15, 15)$. The entropy rate (green, upper) is not monotonically increasing in β.

Varying Parameters

We now consider examples for the landscape derived from the three-type game matrix:

$$G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1}$$

This landscape typically has several local extrema. Let the population size $N' = 6N$. Then we have extrema at the simplex corners $(6N, 0, 0), (0, 6N, 0), (0, 0, 6N)$, center of the boundary simplices $(3N, 3N, 0), (3N, 0, 3N), (0, 3N, 3N)$, and the center $(2N, 2N, 2N)$.

Varying Parameters

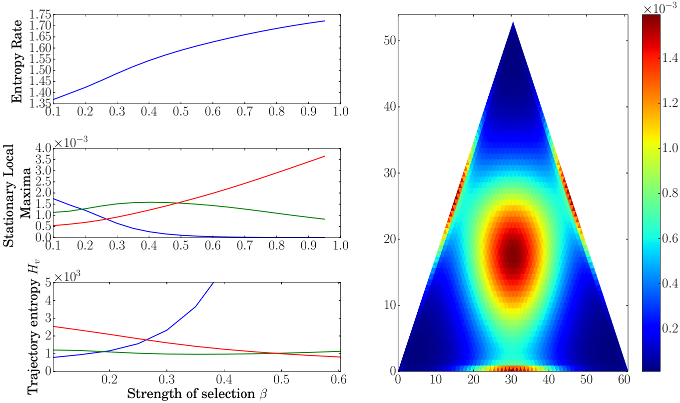

Figure : This $n = 3$ player example with $N = 60$, $\mu = 1/N$ has multiple local stationary extrema, at the center of the simplex, on the centers of the boundary simplices, and on the corners of the simplex. Right: Stationary distribution for $\beta = 0.5$. Left: The relative stability of equilibria changes with β.

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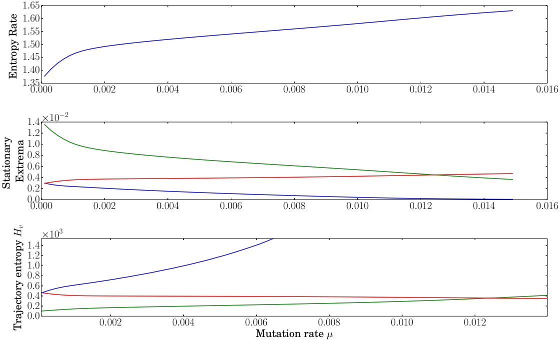

Figure : Same process as before with $\beta = 1$ and varying rate of mutation μ. The value of μ can determine which of the equilibria is most stable. As $\mu \to 0$ the corner states are favored. As μ increases, the interior equilibrium becomes more stable.

Thanks!

Stationary Stability for Evolutionary Dynamics in Finite Populations, Marc Harper, Dashiell Fryer, ArXiv:1311.0941

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